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On potential energy density in an incompressible, stratified fluid

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Exact, positive definite expressions both in power series and in closed form are derived for the potential energy density in a continuously stratified incompressible fluid. They are useful for determining the energy of large displacements or mixing processes in regions of rapidly varying buoyancy frequency, such as a sharp pycnocline.

1. Introduction

A simple expression for potential energy density in a vertically stratified incompressible fluid is well known for the case of linearized fluid motion [see, for example, the books by Lighthill (1978), Turner (1973) and Yih (1965)]. Two versions of he expression are

$$E_p \simeq -\tfrac{1}{2}g\rho_0'(z)\,\zeta^2 \simeq -\frac{1}{2}\frac{g}{\rho_0'(z)}\rho_e^2, \tag{1.1}$$

where ζ is the vertical displacement of a fluid particle, $\rho_0(z)$ is the undisturbed density, $\rho_0'(z)$ is its derivative with respect to height z, and ρ_e is the local deviation of the actual density from ρ_0, i.e.

$$\rho_e = \rho - \rho_0. \tag{1.2}$$

There will be circumstances, for example $\rho_0'(z) \ll \zeta\rho_0''(z)$, where (1.1) is a poor approximation, and where a more accurate expression is needed to determine the energy changes associated with large displacements or mixing processes in a stratified fluid. In this note, exact expressions in both series and closed form are derived for E_p. Although the derivation is straightforward, and other workers are likely to have obtained similar results, the exact theory has not previously been noted in the literature as far as we are aware.

An interesting feature is that the results do not (in contrast to the usual theory of 'available potential energy') depend on any 'containment' assumption: that is, the fluid need not be imagined to be contained within some (finite or infinite) fixed domain, across the boundaries of which there is no mass flux.

In the following paper (Andrews 1981) it is shown that similar results hold for a compressible fluid.

2. Derivation

The equations describing motion of an incompressible, vertically stratified ideal fluid under gravity $\mathbf{g}$ may be written

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p_e + \mathbf{g}\rho_e, \tag{2.1}$$

$$\frac{D\rho}{Dt} = 0, \tag{2.2}$$

$$\nabla . \mathbf{u} = 0, \tag{2.3}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u}.\nabla$, $\mathbf{g} = (0, 0, -g)$ (assumed constant for the moment), and the excess pressure

$$p_e = p - p_0(z), \tag{2.4}$$

where z is the vertical co-ordinate in a Cartesian co-ordinate system (x, y, z) and where

$$\frac{dp_0}{dz} = -g\rho_0(z). \tag{2.5}$$

Scalar multiplication of (2.1) by $\mathbf{u}$ and use of (2.2) and (2.3) yields

$$\frac{D}{Dt} E_k + gw\rho_e + \nabla . (p_e \mathbf{u}) = 0, \tag{2.6}$$

where w is the vertical component of $\mathbf{u}$, and where the kinetic energy density

$$E_k \equiv \tfrac{1}{2}\rho|\mathbf{u}|^2.$$

We can develop the most useful notion of potential energy for these equations by imagining that the fluid motion is set up from the undisturbed state $\mathbf{u} = 0$, $\rho = \rho_0(z)$, $p = p_0(z)$. Suppose that the fluid element at $(\mathbf{x}, t)$ has moved a vertical distance $\zeta(\mathbf{x}, t)$ from its original, undisturbed position. Then

$$D\zeta/Dt = Dz/Dt = w. \tag{2.7}$$

Also (2.2) has the solution

$$\rho(\mathbf{x}, t) = \rho_0(z - \zeta), \tag{2.8}$$

whence

$$gw\rho_e = gw(\rho - \rho_0) = gw\{\rho_0(z-\zeta) - \rho_0(z)\}$$

$$= gw \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \rho_0^{(n)}(z)\, \zeta^n, \tag{2.9}$$

$\rho_0^{(n)}$ being the nth derivative of $\rho_0(z)$. It can now be verified using (2.7) and the chain rule that the right-hand side of (2.9) is equal to DE_p/Dt, where E_p is the following function of z and ζ:

$$E_p = g \sum_{n=1}^{\infty} \frac{n}{(n+1)!} (-1)^n \rho_0^{(n)}(z)\, \zeta^{n+1}. \tag{2.10}$$

Hence (2.6) becomes

$$\frac{D}{Dt}(E_k + E_p) + \nabla . (p_e \mathbf{u}) = 0, \tag{2.11}$$

an exact expression of energy conservation in the desired form. We note that (1.1) is just the leading term of (2.10).

The series (2.10) can evidently be summed to give

$$E_p = I_1(z,\zeta) \equiv -\int_0^{\zeta} g\tilde{\zeta}\rho_0'(z-\tilde{\zeta})\,d\tilde{\zeta}, \tag{2.12}$$

exhibiting the positive-definite character of E_p when the undisturbed stratification is stable everywhere:

$$-g\rho_0'(z) > 0, \quad \forall z. \tag{2.13}$$

An alternative closed-form expression for E_p can be found when (2.13) holds, which for some purposes is more elegant since it expresses E_p as a non-negative function of ρ_0 and ρ_e, rather than of z and ζ. This is possible because by (2.8) ρ contains all the Lagrangian information relevant to (2.12), when (2.13) holds.† Define the function $\chi\{\,\}$ by

$$\chi\{\rho_0(z)\} = gz; \tag{2.14}$$

this function is a property of the undisturbed state and is a monotonically decreasing function of its argument when (2.13) holds. Then

$$E_p = I_2(\rho_0,\rho_e) \equiv -\int_0^{\rho_e} [\chi\{\rho_0(z)+\tilde{\rho}_e\} - \chi\{\rho_0(z)\}]\,d\tilde{\rho}_e. \tag{2.15}$$

The equivalence of (2.15) with (2.12) follows by noting from (2.14) that

$$\chi\{\rho_0(z-\zeta)\} - \chi\{\rho_0(z)\} = -g\zeta,$$

or

$$\chi\{\rho_0(z)+\rho_e\} - \chi\{\rho_0(z)\} = -g\zeta, \tag{2.16}$$

since

$$\rho_0(z)+\rho_e = \rho_0(z-\zeta) \tag{2.17}$$

by (2.8) and (1.2). At constant z,

$$d\rho_e = -\rho_0'(z-\zeta)\,d\zeta; \tag{2.18}$$

(2.16) and (2.18), with the dummy variables of integration $\tilde{\zeta}$ and $\tilde{\rho}_e$ substituted for ζ and ρ_e, show that the two integrals (2.12) and (2.15) are equal. It is easy to check that (2.15) reduces to the last expression in (1.1) for small ρ_e.

As a simple example when ρ_e is not small, take the case of constant buoyancy (Brunt–Väisälä) frequency N, i.e.

$$\rho_0(z) = C\exp\left(-\frac{N^2}{g}z\right), \tag{2.19}$$

Then we obtain from (2.15) the exact formula

$$E_p = \frac{g^2}{N^2}\left\{(\rho_0+\rho_e)\ln\left(1+\frac{\rho_e}{\rho_0}\right) - \rho_e\right\}. \tag{2.20}$$

The form (2.15) of our general result is valid, almost as it stands, when the gravitational field is non-uniform. We need only replace gz on the right of (2.14) by the gravitational potential $\Phi(\mathbf{x})$, and express ρ_0 and p_0 as functions of Φ.

† We may actually replace $>$ in (2.13) by $\geqslant$; a full discussion is given in McIntyre (1981).

3. Discussion

Integrating (2.12) by parts and using (2.5) gives

$$E_p = g\zeta\rho_0(z-\zeta)+p_0(z)-p_0(z-\zeta). \tag{3.1}$$

This form is less useful in practice than the explicitly non-negative forms (2.12) and (2.15), but it has a simple physical interpretation and so makes it clearer, in turn, why the results of § 2 are so simple. Consider a small unit volume of fluid that has been moved quasi-statically from $z-\zeta$ to z by some external agency. The agency must do work $g\zeta\rho_0(z-\zeta)$ to overcome the external gravitational force. However, we should also include the work done against the basic pressure-gradient force, $-\nabla p_0$, which is responsible for equilibrium. That work is $p_0(z)-p_0(z-\zeta)$, and so the net work required is just the right-hand side of (3.1).

Equation (2.10) shows that (1.1) is exact only if the fluid is linearly stratified. In other cases the accuracy of (1.1) depends on how rapidly $\rho_0'(z)$ changes. A common situation where (1.1) can make a significant error is in mixing processes or large vertical displacements that occur in the ocean at or near a very sharp pycnocline.

Numerical analyses of problems in stratified flow frequently utilize the total potential energy

$$P_{\text{tot}} = \int gz\rho\, dV,$$

(see, for example, Warn-Varnas *et al.* 1979); the integral is taken over a domain bounded by fixed surfaces through which there is no mass flow. The relationship with our results can be seen as follows. Using (2.2), (2.3), and the identity

$$d\left(\int \phi\, dV\right)/dt = \int (D\phi/Dt)\, dV$$

(which itself depends on (2.3) as well as on the assumed boundary conditions), we obtain

$$\frac{dP_{\text{tot}}}{dt} = \int gw\rho\, dV = \int gw\rho_e\, dV = \frac{d}{dt}\int E_p\, dV, \tag{3.2}$$

where the last step uses (2.9) *et seq.*, and the second last step the fact that the horizontal area integral of w vanishes at each z, another consequence of (2.3) and the boundary condition of no mass flow. The relation (3.2) verifies the equivalence, in the circumstances assumed, between the conventional exact expression for potential energy changes in the whole fluid and the exact expression $\int E_p\, dV$ derived from the conservation relation (2.11).

The relation between our results and the concept of 'available potential energy' (APE) used in meteorology is of interest. The APE would be defined in the present context as

$$\text{APE} = P_{\text{tot}} - P^0_{\text{tot}}, \tag{3.3}$$

where P^0_{tot} denotes the value of P_{tot} for the undisturbed state and where a domain with fixed, impermeable boundaries is again assumed (Lorenz 1955). The definition (3.3) implies that the APE is zero for the undisturbed state, whence it follows from (3.2) that

$$\text{APE} = \int E_p\, dV. \tag{3.4}$$

However the results of § 2 are more general in the sense that, whereas the APE is defined only globally, E_p is defined locally by (2.15), is governed by the local conservation relation (2.11), and is independent of any assumptions about the boundary conditions.

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